Dyonic giant magnons

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ABSTRACT: We study the classical spectrum of string theory on $AdS_5 \times S^5$ in the Hofman-Maldacena limit. We find a family of classical solutions corresponding to Giant Magnons with two independent angular momenta on S^5. These solutions are related via Pohlmeyer's reduction procedure to the charged solitons of the Complex sine-Gordon equation. The corresponding string states are dual to BPS boundstates of many magnons in the spinchain description of planar $\mathcal{N} = 4$ SUSY Yang-Mills. The exact dispersion relation for these states is obtained from a purely classical calculation in string theory.

The AdS/CFT correspondence predicts that the spectrum of operator dimensions in planar $\mathcal{N} = 4$ SUSY Yang-Mills and the spectrum of a free string on $AdS_5 \times S^5$ are the same. Verifying this prediction by computing the full spectrum is an important unsolved problem. For small 't Hooft coupling, $\lambda \ll 1$, the perturbative dimensions of gauge theory operators can be calculated by diagonalising an integrable spin chain [1, 2]. For $\lambda \gg 1$, the string sigma-model, which is also classically integrable [3], becomes weakly coupled. The full spectrum certainly depends in a complicated way on the 't Hooft coupling. However the appearance of integrability on both sides of the correspondence strongly suggests that the problem is solvable and there has been significant progress in formulating exact Bethe ansatz equations which hold for all values of the coupling [4−8].

In recent work, Hofman and Maldacena (HM) [9] have identified a particular limit where the problem of determining the spectrum simplifies considerably. In this limit we restrict our attention to Yang-Mills operators with large U(1) R-charge J_1, which therefore also have large scaling dimensions $\Delta \geq J_1$. More precisely, HM consider a limit where Δ and J_1 become infinite with the difference $\Delta - J_1$ and the 't Hooft coupling λ held fixed (see also [10]). In this limit both the gauge theory spin chain and the dual string effectively become infinitely long. The spectrum can then be analysed in terms of asymptotic states and their scattering. On both sides of the correspondence the limiting theory is characterised by a centrally-extended SU(2|2) × SU(2|2) supergroup which strongly constrains the spectrum and S-matrix [8].

The basic asymptotic state carries a conserved momentum p, and lies in a short multiplet of supersymmetry. States in this multiplet have different polarisations corresponding to transverse fluctuations of the dual string in different directions in $AdS_5 \times S^5$. The BPS condition essentially determines the dispersion relation for all these states to be [4, 6−8] (see also [11]),

$$\Delta - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right)}. \tag{1}$$

In the spin chain description, this multiplet corresponds to an elementary excitation of the ferromagnetic vacuum known as a magnon. The dual state in semiclassical string theory was identified in [9]. It corresponds to a localised classical soliton which propagates on an infinite string moving on an $\mathbb{R} \times S^2$ subspace of $AdS_5 \times S^5$. The conserved magnon momentum p corresponds to a certain geometrical angle in the target space explaining the periodic momentum dependence appearing in (1). Following [9], we will refer to this classical string configuration as a Giant Magnon.

In addition to the elementary magnon, the asymptotic spectrum of the spin chain also contains an infinite tower of boundstates [14]. Magnons with polarisations in an SU(2) subsector carry a second conserved U(1) R-charge, denoted J_2, and form boundstates with the exact dispersion relation,

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right)}. \tag{2}$$

The elementary magnon in this subsector has charge $J_2 = 1$ and states with $J_2 = Q$ correspond to Q-magnon boundstates. These states should exist for all integer values of

J_2 and for all values of the 't Hooft coupling [14]. In particular we are free to consider states where $J_2 \sim \sqrt{\lambda}$. For such states the dispersion relation (2) has the appropriate scaling for a classical string carrying a second large classical angular momentum J_2. In this Letter we will identify the corresponding classical solutions of the worldsheet theory and determine some of their properties. In particular, we will reproduce the exact BPS dispersion relation (2) from a purely classical calculation in string theory. In the case of boundstates with momentum $p = \pi$, the relevant configuration was already obtained in [14] as a limit of the two-spin folded string solution of [15, 16]. Here we will obtain the general solution for arbitrary momentum p and R-charge J_2. Both the $p = \pi$ configuration of [14] and the original single-charge Giant Magnon of HM will emerge as special cases. We also briefly discuss semiclassical quantisation of these objects which simply has the effect of restricting the R-charge J_2 to integer values.

The minimal string solutions carrying two independent angular momenta, J_1 and J_2 correspond to strings moving on an $\mathbb{R} \times S^3$ subspace of $AdS_5 \times S^5$. In static gauge, the string equations of motion are essentially those of a bosonic $O(4)$ σ-model supplemented by the Virasoro constraints. An efficient way to find the relevant classical solutions exploits the equivalence of this system to the Complex sine-Gordon (CsG) equation discovered many years ago by Pohlmeyer [12][1]. The CsG equation is completely integrable and has a family of soliton solutions [17 – 19]. In addition to a conserved momentum the soliton also carries an additional conserved charge associated with rotations in an internal space. Each CsG soliton corresponds to a classical solution for the string. The problem of reconstructing the corresponding string motion, while still non-trivial, involves solving linear differential equations only. We construct the two-parameter family of string solutions corresponding to a single CsG soliton and show that they have all the expected properties of Giant Magnons. In particular they carry non-zero J_2 and obey the BPS dispersion relation (2). It is quite striking that we obtain the *exact* BPS formula, for all values of J_2, from a classical calculation. This situation seems to be very analogous to that of BPS-saturated Julia-Zee dyons in $\mathcal{N} = 4$ SUSY Yang-Mills [20, 21]. These objects also have a classical BPS mass formula which turns out to be exact. It seems appropriate to call our new two-charge configurations Dyonic Giant Magnons.

Multi-soliton solutions of the CsG equation are also available in the literature [17 – 19]. In the classical theory these objects undergo factorised scattering with a known time-delay. This is precisely the information required to calculate the semiclassical approximation to the S-matrix and we hope to return to this in the near future. The rest of the paper is organised as follows. We begin by discussing strings on $\mathbb{R} \times S^3$ in the HM limit and review the original Giant Magnon solution of [9] in this context. We then explain the classical equivalence of the string equations to the complex sine-Gordon equation. Finally, we construct the required string solutions from the CsG solitons and determine their properties.

We will begin by focusing on closed bosonic strings moving on an $\mathbb{R} \times S^3$ subspace of $AdS_5 \times S^5$. The worldsheet coordinates are denoted $\sigma \sim \sigma + 2\pi$ and $-\infty < \tau < \infty$ while

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[1]The corresponding reduction of the $O(3)$ sigma model to the ordinary sine-Gordon equation [12] was discussed in [9] (see also [13]).

those on the target space are,

$$X_0(\sigma, \tau), \quad \vec{X}(\sigma, \tau) = (X_1, X_2, X_3, X_4) \quad \text{with} \ |\vec{X}|^2 = 1. \tag{3}$$

We make the static gauge choice $X_0(\sigma, \tau) = \kappa \tau$, so that the string energy is given as $\Delta = \sqrt{\lambda}\,\kappa$. We define a $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ Cartan subgroup of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry group of the target space under which the complex coordinates $Z_1 = X_1 + iX_2$ and $Z_2 = X_3 + iX_4$ have charges $(1, 0)$ and $(0, 1)$ respectively. String states carry the corresponding conserved Noether charges,

$$J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, \mathrm{Im}[\bar{Z}_1 \partial_\tau Z_1], \tag{4}$$

$$J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \, \mathrm{Im}[\bar{Z}_2 \partial_\tau Z_2], \tag{5}$$

which can be thought of as angular momenta in two orthogonal planes within S^3.

We will now describe the HM limit [9] where one of these angular momenta, say J_1, becomes large while the other, J_2, is held fixed. The specific limit we consider is,

$$\begin{aligned} J_1 &\to \infty, \quad \Delta \to \infty \\ \Delta - J_1 &= \text{fixed}, \quad \lambda = \text{fixed}, \quad J_2 = \text{fixed}. \end{aligned} \tag{6}$$

The fact that λ is held fixed allows us to interpolate between the regimes of small and large λ, where perturbative gauge theory and semiclassical string theory respectively are valid. It is convenient to implement the HM limit for the string by defining the following rescaled worldsheet coordinates, $(x, t) \equiv (\kappa \sigma, \kappa \tau)$, which are held fixed as $\kappa = \Delta / \sqrt{\lambda} \to \infty$. Under this rescaling, the interval $-\pi \le \sigma \le \pi$ corresponding to the closed string is mapped to the real line $-\infty \le x \le \infty$ with the point $\sigma = \pm \pi$ mapped to $x = \pm\infty$.

As always, a consistent closed string configurations always involve at least two magnons with zero total momentum. As explained in [9], magnon momentum is associated with a certain geometrical angle in the target space. The condition that the total momentum vanishes (modulo 2π) is then enforced by the closed string boundary condition. However, after the above rescaling, closed string boundary condition and thus the vanishing of the total momentum can actually be relaxed. This allows us to focus on a single worldsheet excitation or magnon carrying non-zero momentum p. This makes sense because the additional magnon with momentum $-p$ required to make the configuration consistent can be hidden at the point $x = \infty$.

The conserved charges of the system which remain finite in the HM limit are given as,

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{+\infty} dx \, (1 - \mathrm{Im}[\bar{Z}_1 \partial_t Z_1]), \tag{7}$$

$$J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} dx \, \mathrm{Im}[\bar{Z}_2 \partial_t Z_2]. \tag{8}$$

It is important to note that these quantities will not necessarily be equal to their counterparts (4), (5) computed in the original worldsheet coordinates. In general, the latter may

include an additional contribution coming from a neighbourhood of the point $\sigma = \pm\pi$ which is mapped to $x = \infty$. As in the above discussion of worldsheet momentum, the extra contribution reflects the presence of additional magnons at infinity.

The equation of the motion for the target space coordinate $\vec{X}(x,t)$ can be written in terms of light-cone coordinates $x_\pm = (t \pm x)/2$ as

$$\partial_+\partial_-\vec{X} + (\partial_+\vec{X} \cdot \partial_-\vec{X})\vec{X} = 0\,. \tag{9}$$

A physical string solution must also satisfy the Virasoro constraints. In terms of the rescaled coordinates these become,

$$\partial_+\vec{X} \cdot \partial_+\vec{X} = \partial_-\vec{X} \cdot \partial_-\vec{X} = 1\,. \tag{10}$$

Before discussing the general two-charge case, we will review the simpler situation where $J_2 = 0$ which leads to the basic Giant Magnon solution of [9]. This corresponds to restricting our attention to strings which are constrained to lie on an S^2 submanifold of S^3. In terms of the worldsheet fields introduced above, we can implement this by setting $X_4(x,t) = 0$ or equivalently demanding that $Z_2(x,t)$ is real. In this case the complex worldsheet fields Z_1 and Z_2 can be written as,

$$Z_1(x,t) = \sin\theta\exp(i\varphi)\,, \qquad Z_2(x,t) = \cos\theta\,, \tag{11}$$

where θ and φ are the polar and azimuthal angles on the two-sphere respectively. In these coordinates the angular momentum J_1 generates shifts of the azimuthal angle φ. The required solution should have infinite energy Δ and angular momentum J_1 with a finite difference $\Delta - J_1$. Such a configuration can be obtained by considering an open string[2] with both endpoints moving on the equator $\theta = \pi/2$ at the speed of light. The string theory quantity corresponding to the magnon momentum p is exactly given by the angular separation $\Delta\varphi$ between these two endpoints of the string (see figure 1). The Giant Magnon solution therefore has the following boundary conditions for Z_1 and Z_2,

$$Z_1 \to \exp\left(it \pm i\frac{p}{2}\right)\,, \qquad Z_2 \to 0\,, \quad \text{as} \ \ x \to \pm\infty\,. \tag{12}$$

In the following we will show that the unique solution with the required properties which satisfies (12) is

$$Z_1 = \left[\sin\left(\frac{p}{2}\right)\tanh(Y) - i\cos\left(\frac{p}{2}\right)\right]\exp(it)\,, \qquad Z_2 = \frac{\sin\left(\frac{p}{2}\right)}{\cosh(Y)}\,, \tag{13}$$

where

$$Y = \frac{x - \cos\left(\frac{p}{2}\right)t}{\sin\left(\frac{p}{2}\right)}\,. \tag{14}$$

This solution is equivalent to the one given as Eqn (2.16) in [9]. We will rederive it as a special case of the more general solution presented below. One may check that while the

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[2]Recall that in the HM limit we have relaxed the closed string boundary condition. To obtain a consistent closed string configuration we should add a second magnon at infinity in the coordinate x. In spacetime this corresponds to adding a second open string to form a folded closed string.
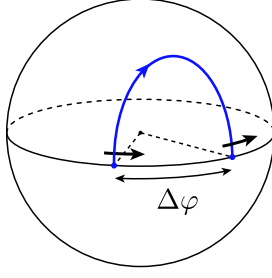
Figure 1: A Giant Magnon solution. The endpoints of the string move on the equator $\theta = \pi/2$ at the speed of light. The magnon momentum is given by $p = \Delta\varphi$, where $\Delta\varphi$ is the angular distance between two endpoints of the string.

energy Δ and the angular momentum J_1 of the solution diverge, the combination $\Delta - J_1$ remains finite and is given by,

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi} \left| \sin\left(\frac{p}{2}\right) \right| \tag{15}$$

in agreement with the large-λ limit of (1). Moreover the solution (13) carries only one non-vanishing angular momentum, having $J_2 = 0$.

To solve the string equations of motion (9) in the general case, together with the Virasoro conditions (10), we will exploit the equivalence of this system with the CsG equation. Following [12], we will begin by identifying the the SO(4) invariant combinations of the worldsheet fields $\vec{X}$ and their derivatives. As the first derivatives $\partial_\pm \vec{X}$ are unit vectors, we can define a real scalar field $\phi(x, t)$ via the relation,

$$\cos \phi = \partial_+ \vec{X} \cdot \partial_- \vec{X} \,. \tag{16}$$

Taking into account the constraint $|\vec{X}|^2 = 1$, we see that there are no other independent SO(4) invariant quantities that can be constructed out of the fields and their first derivatives. At the level of second derivatives we can construct two additional invariants;

$$u \sin \phi = \partial_+^2 \vec{X} \cdot \vec{K} \,, \quad v \sin \phi = \partial_-^2 \vec{X} \cdot \vec{K} \,, \tag{17}$$

where the components of vector $\vec{K}$ are given by $K_i = \epsilon_{ijkl} X_j \partial_+ X_k \partial_- X_l$. The connection with CsG model arises from the equations of motion for u, v and ϕ derived in [12]. In fact the resulting equations imply that u and v are not independent and can be eliminated in favour of a new field $\chi(x, t)$ as,

$$u = \partial_+ \chi \tan\left(\frac{\phi}{2}\right) \,, \quad v = -\partial_- \chi \tan\left(\frac{\phi}{2}\right) \,. \tag{18}$$

The equations of motion for χ and ϕ can then be written as

$$\partial_+ \partial_- \phi + \sin \phi - \frac{\tan^2\left(\frac{\phi}{2}\right)}{\sin \phi} \partial_+ \chi \partial_- \chi = 0 \,, \tag{19}$$

$$\partial_+ \partial_- \chi + \frac{1}{\sin \phi}(\partial_+ \phi \partial_- \chi + \partial_- \phi \partial_+ \chi) = 0 \,. \tag{20}$$

In the special case of constant χ they reduce to the usual sine-Gordon equation for $\phi(x,t)$. Finally we can combine the real fields ϕ and χ to form a complex field $\psi = \sin(\phi/2)\exp(i\chi/2)$, which obeys the equation,

$$\partial_+\partial_-\psi + \psi^* \frac{\partial_+\psi\partial_-\psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0\,. \tag{21}$$

Equation (21) is known as the Complex sine-Gordon equation. Like the ordinary sG equation, it is completely integrable and has localised soliton solutions which undergo factorised scattering. The CsG equation is invariant under a global rotation of the phase of the complex field: $\psi \to \exp(i\nu)\psi$, $\psi^* \to \exp(-i\nu)\psi^*$. In addition to momentum and energy, CsG solitons carry the corresponding conserved U(1) Noether charge[3]. The most general one soliton solution to (21) is given by (see eg [18]),

$$\psi_{1\text{-soliton}} = e^{i\mu}\frac{\cos(\alpha)\exp(i\sin(\alpha)T)}{\cosh(\cos(\alpha)(X-X_0))} \tag{22}$$

with

$$X = \cosh(\theta)x - \sinh(\theta)t\,, \qquad T = \cosh(\theta)t - \sinh(\theta)x\,. \tag{23}$$

The constant phase μ is irrelevant for our purposes as only the derivatives of the field χ affect the corresponding string solution. The parameter X_0 can be absorbed by a constant translation of the world-sheet coordinate x and we will set it to zero. The two remaining parameters of the solution are the rapidity θ of the soliton and an additional real number α which determines the U(1) charge carried by the soliton.

Taking the limit $\alpha \to 0$, the field ϕ corresponding to the one-soliton solution (22) reduces to the kink solution of the ordinary sG equation. As it is the only known solution of the CsG equation with this property, it is the unique candidate for the dyonic Giant Magnon solution we seek. It remains to reconstruct the corresponding configuration of the string worldsheet fields $\vec{X}$ (or equivalently Z_1 and Z_2) corresponding to (22) for general values of the rapidity θ and rotation parameter α.

In this case we have,

$$\partial_+\vec{X} \cdot \partial_-\vec{X} = \cos(\phi) = 1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)}\,. \tag{24}$$

Hence the complex coordinates Z_1 and Z_2 must both solve the linear equation,

$$\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial x^2} + \left[1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)}\right]Z = 0 \tag{25}$$

where, as above $X = \cosh(\theta)x - \sinh(\theta)t$. We impose the boundary conditions at $x \to \pm\infty$ appropriate for a Giant Magnon with momentum p,

$$Z_1 \to \exp\left(it \pm i\frac{p}{2}\right)\,, \tag{26a}$$

$$Z_2 \to 0\,. \tag{26b}$$

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[3]Note that there is no simple relation between the U(1) charge of the CsG soliton and the string angular momentum J_2.

As always the two complex fields obey the constraint $|Z_1|^2 + |Z_2|^2 = 1$. We will find unique solutions of the linear equation (25) obeying these conditions and then, for self-consistency, check that they correctly reproduce (24).

It is convenient to express the solution of (25) in terms of the boosted coordinates X and T. In terms of these variables $Z = Z[X, T]$ obeys,

$$\frac{\partial^2 Z}{\partial T^2} - \frac{\partial^2 Z}{\partial X^2} + \left[1 - \frac{2\cos^2(\alpha)}{\cosh^2(\cos(\alpha)X)} \right] Z = 0 \,. \tag{27}$$

The problem now has the form of a Klein-Gordon equation describing the scattering of a relativistic particle in one spatial dimension incident on a static potential well. As usual the general solution of this equation can be written as a linear combination of "stationary states" of the form,

$$Z_\omega = F_\omega(X) \exp(i\omega T) \,. \tag{28}$$

Rescaling the variables according to,

$$\xi = \cos(\alpha)X \,, \quad f(\xi) = F_\omega(X) \,, \quad \varepsilon = \frac{\sqrt{\omega^2 - 1}}{\cos(\alpha)} \,, \tag{29}$$

we find that the function $f(\xi)$ obeys the equation,

$$-\frac{d^2 f}{d\xi^2} - \frac{2}{\cosh^2(\xi)} f = \varepsilon^2 f \,. \tag{30}$$

Equation (30) coincides with the time-independent Schrödinger equation for a particle in (a special case of) the Rosen-Morse potential [23],

$$V(\xi) = \frac{-2}{\cosh^2(\xi)} \,. \tag{31}$$

The exact spectrum of this problem is known (see *e.g.* [24]). There is a single normalisable boundstate with energy $\varepsilon^2 = -1$ and wavefunction,

$$f_{-1}(\xi) = \frac{1}{\cosh(\xi)} \tag{32}$$

and a continuum of scattering states with $\varepsilon^2 = k^2$ for $k > 0$ and wavefunctions,

$$f_{k^2}(\xi) = \exp(ik\xi)\left(\tanh(\xi) - ik\right) \tag{33}$$

with asymptotics,

$$f_{k^2}(\xi) \rightarrow \exp\left(ik\xi \pm i\frac{\delta}{2}\right) \tag{34}$$

where the scattering phase-shift is given as $\delta = 2\tan^{-1}(1/k)$.

The general solution to the original linear equation (25) can be constructed as a linear combination of these boundstate and scattering wavefunctions. The particular solutions

corresponding to the worldsheet fields Z_1 and Z_2 are singled out by the boundary conditions (26a), (26b). In particular, the boundary condition (26a) can only be matched by a solution corresponding to a single scattering mode $f_{k^2}(\xi)$;

$$Z_1 = c_1 f_{k^2} \left(\cos(\alpha)X\right) \exp(i\omega_{k^2}T) \tag{35}$$

where $\omega_{k^2} = \sqrt{k^2 \cos^2(\alpha) + 1}$. We find that (26a) is obeyed provided we set,

$$k = \frac{\sinh(\theta)}{\cos(\alpha)} \tag{36}$$

which yields the magnon momentum $p = \delta = 2\tan^{-1}(1/k)$. The boundary condition (26b) dictates that Z_2 decays at left and right infinity. This is only possible if we identify it with the solution corresponding to the unique normalisable boundstate of the potential (30),

$$Z_2 = c_2 f_{-1}(\cos(\alpha)X) \exp(i\omega_{-1}T) \tag{37}$$

with $\omega_{-1} = \sin(\alpha)$. Without loss of generality we can choose the constants c_1 and c_2 to be real. The condition $|Z_1|^2 + |Z_2|^2 = 1$ then yields,

$$c_1 = c_2 = \frac{1}{\sqrt{1+k^2}}. \tag{38}$$

To summarise the above discussion the resulting string solution is,

$$Z_1 = \frac{1}{\sqrt{1+k^2}} \left(\tanh\left[\cos(\alpha)X\right] - ik\right) \exp(it),$$
$$Z_2 = \frac{1}{\sqrt{1+k^2}} \frac{1}{\cosh\left[\cos(\alpha)X\right]} \exp\left(i\sin(\alpha)T\right), \tag{39}$$

where X, T and k are defined in (23) and (36) above. One may easily check that this solution, in addition to obeying the string equation of motion (25) and boundary conditions (26a), (26b), obeys the Virasoro constraints and satisfies the self-consistency condition (24). It also reduces to the Hofman-Maldacena solution (13) in the non-rotating case $\alpha = 0$. Setting $p = \pi$, we obtain one-half[4] of the folded string configuration discussed in [14].

The solution (39) depends on two parameters: k and α. We can now evaluate the conserved charges (7) and (8) as a function of these parameters,

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi} \frac{1}{1+k^2} \frac{\sqrt{1 + k^2 \cos^2(\alpha)}}{\cos(\alpha)},$$
$$J_2 = \frac{\sqrt{\lambda}}{\pi} \frac{1}{1+k^2} \tan(\alpha). \tag{40}$$

As above the magnon momentum is identified as $p = 2\tan^{-1}(1/k)$. Eliminating k and α we obtain the dispersion relation,

$$\Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}, \tag{41}$$

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[4]See footnote below Eqn (11).

which agrees precisely with the BPS dispersion relation (2) for the magnon boundstates obtained in [14].

The time dependence of the solution (39) is also of interest. As in the orginal HM solution the constant phase rotation of Z_1 with exponent it ensures that the endpoints of the string move on an equator of the three-sphere at the speed of light. We can remove this dependence by changing coordinates from Z_1 to $\widetilde{Z}_1 = \exp(-it)Z_1$. In the new frame, the string configuration depends periodically on time through the t-dependence of Z_2. The period, $\mathcal{T}$, for this motion is the time for the solution to come back to itself up to a translation of the worldsheet coordinate x. From (39) we find,

$$\mathcal{T} = 2\pi \frac{\cosh(\theta)}{\sin(\alpha)} \,. \tag{42}$$

As we have a periodic classical solution it is natural to define a corresponding action variable. A leading-order semiclassical quantization can then be performed by restricting the action variable to integral values according to the Bohr-Sommerfeld condition. Following [9], the action variable I is defined by the equation,

$$dI = \frac{\mathcal{T}}{2\pi} \ d(\Delta - J_1)|_p \,. \tag{43}$$

where the subscript p indicates that the differential is taken with fixed p. Using (40), (41) and (42) we obtain simply $dI = dJ_2$ which is consistent with the identification $I = J_2$. This is very natural as we expect the angular momentum J_2 to be integer valued in the quantum theory. It is also consistent with the semiclassical quantization of finite-gap solutions discussed in [25] where the action variables correspond to the filling fractions. These quantities are simply the number of units of J_2 carried by each worldsheet excitation.

Acknowledgments

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